

## Three useful theorems

The following hold true for any meaningful expressions  $A$  and  $B$ .

- **Thm 1.**  $|A| = B \implies A = \pm B$ , providing  $B$  is not negative
- **Thm 2.**  $|A| \leq B \implies -B \leq A \leq B$ , providing  $B$  is not negative.
- **Thm 3.**  $|A| \geq B \implies A \leq -B$  or  $A \geq B$

## Examples

### ■ Illustrating use of Thm 1

[EX1]  $|x| = 3$

[EX2]  $|3x + 7| = 2x + 8$

[EX3]  $|3x + 7| = 2x - 8$

[EX4]  $|3x + 7| = 2x + \frac{14}{3}$

### ■ Illustrating use of Thm 2

[EX1]  $|x| < 3$

[EX2]  $|3x| < -8$

[EX3]  $|3x + 7| < 2x + 8$

[EX4]  $|3x + 7| < 2x - 8$

### ■ Illustrating use of Thm 3

[EX4]  $|3x + 5| > 4$

[EX5]  $|3x + 5| > -4$

## Proofs

The following hold true for any meaningful expressions  $A$  and  $B$ .

- $|A| = B \implies A = \pm B$ , providing  $B$  is not negative
- $|A| < B \implies -B < A < B$ , providing  $B$  is not negative
- $|A| > B \implies A < -B$  or  $A > B$

## Three useful theorems

The following hold true for any meaningful expressions  $A$  and  $B$ .

■ **Thm 1.**  $|A| = B \Rightarrow A = \pm B$ , providing  $B$  is not negative

$|A| = B$  says  $A$  is exactly  $B$  units from zero.



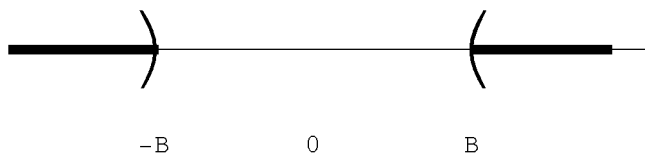
■ **Thm 2.**  $|A| \leq B \Rightarrow -B \leq A \leq B$ , providing  $B$  is not negative.

$|A| < B$  says  $A$  is within  $B$  units of zero.



■ **Thm 3.**  $|A| \geq B \Rightarrow A \leq -B$  or  $A \geq B$

$|A| > B$  says  $A$  is farther than  $B$  units from zero.



# Examples

## ■ Illustrating use of Thm 1

$$[\text{EX1}] \quad |x| = 3$$

Since  $3 \geq 0$ , Theorem 1 may be used.

$$\therefore x \in \{-3, 3\}$$

$$[\text{EX2}] \quad |3x + 7| = 2x + 8$$

Require  $2x + 8 \geq 0$ . Then  $x \geq -4$ .

$$\text{Case 1: } 3x + 7 = 2x + 8 \implies x = 1$$

**OR**

$$\text{Case 2: } 3x + 7 = -(2x + 8) \implies x = -3$$

A solution must also satisfy the condition  $x \geq -4$ . Both 1 and  $-3$  meet this condition, so neither is eliminated.

$$\therefore x = \{1, -3\}.$$

$$[\text{EX3}] \quad |3x + 7| = 2x - 8$$

Require  $2x - 8 \geq 0$ . Then  $x \geq 4$ .

$$\text{Case 1: } 3x + 7 = 2x - 8 \implies x = -15$$

**OR**

$$\text{Case 2: } 3x + 7 = -(2x - 8) \implies x = \frac{1}{5}$$

A solution must also satisfy the condition  $x \geq 4$ . Neither  $-15$  nor  $\frac{1}{5}$  meet this condition, so each is eliminated.

$$\therefore x \in \emptyset.$$

$$[\text{EX4}] \quad |3x + 7| = 2x + \frac{14}{3}$$

Require  $2x + \frac{14}{3} \geq 0$ . Then  $x \geq \frac{-7}{3}$ .

$$\text{Case 1: } 3x + 7 = 2x + \frac{14}{3} \implies x = \frac{-7}{3}$$

**OR**

$$\text{Case 2: } 3x + 7 = -(2x + \frac{14}{3}) \implies x = \frac{-7}{3}$$

The condition  $x \geq \frac{-7}{3}$  is satisfied, so

$$\therefore x = \frac{-7}{3}.$$

■ Illustrating use of Thm 2

[EX1]  $|x| < 3$

Solution

$$-3 < x < 3$$

$$\therefore x \in (-3, 3)$$

[EX2]  $|3x| < -8$

By definition, the absolute value of a number must be nonnegative, this inequality has no solution.

$$\therefore x \in \emptyset.$$

[EX3]  $|3x + 7| < 2x + 8$

Solution

Require that  $2x + 8 \geq 0$ . Then  $-4 \leq x$ .

$$-(2x + 8) < 3x + 7 < 2x + 8$$

Ineq 1:

$$-(2x + 8) < 3x + 7$$

$$\implies x > -3$$

$$-4 \leq x \wedge -3 < x$$

$$x \in (-3, \infty +)$$

**AND**

Ineq 2:

$$3x + 7 < 2x + 8$$

$$\implies x < 1$$

$$-4 \leq x \wedge x < 1$$

$$x \in [-4, 1)$$

$$x \in [-4, 1) \cap (-3, \infty +)$$

$$\therefore x \in (-3, 1)$$

$$[\text{EX4}] \quad |3x + 7| < 2x - 8$$

Solution

Require that  $2x - 8 \geq 0$ . Then  $4 \leq x$ .

$$-(2x - 8) < 3x + 7 < 2x - 8$$

Ineq 1:

$$\begin{aligned} -(2x - 8) &< 3x + 7 \\ \implies x &> \frac{1}{5} \end{aligned}$$

$$\begin{aligned} 4 \leq x \wedge \frac{1}{5} &< x \\ x &\in [4, \infty +) \end{aligned}$$

**AND**

Ineq 2:

$$\begin{aligned} 3x + 7 &< 2x - 8 \\ \implies x &< -15 \end{aligned}$$

$$\begin{aligned} 4 \leq x \wedge x &< -15 \\ x &\in \emptyset \end{aligned}$$

$$x \in [4, \infty +) \cap \emptyset$$

$$\therefore x \in \emptyset$$

■ Illustrating use of Thm 3

[EX1]  $|3x + 5| > 4$

Solution

$$3x + 5 < -4 \vee 3x + 5 > 4$$

Ineq 1:

$$3x + 5 < -4$$

$$\implies x < -3$$

$$x \in (-\infty, -3)$$

**OR**

Ineq 2:

$$3x + 5 > 4$$

$$\implies x > -\frac{1}{3}$$

$$x \in (-\frac{1}{3}, \infty +)$$

$$\therefore x \in (-\infty, -3) \cup (-\frac{1}{3}, \infty +)$$

$$[\text{EX2}] \quad |3x + 5| > -4$$

Solution

$$3x + 5 < 4 \vee 3x + 5 > -4$$

Ineq 1:

$$3x + 5 < 4$$

$$\implies x < -\frac{1}{3}$$

$$x \in (-\infty, -\frac{1}{3})$$

**OR**

Ineq 2:

$$3x + 5 > -4$$

$$\implies x > -3$$

$$x \in (-3, \infty +)$$

$$x \in (-\infty, -\frac{1}{3}) \cup (-3, \infty +)$$

$$\therefore x \in \mathbb{R}$$

*Note: We should have known this solution by inspecting  $|3x + 5| > -4$ . The absolute value of every number is greater than any negative number.*

# Proofs

The following hold true for any meaningful expressions  $A$  and  $B$ .

■  $|A| = B \implies A = \pm B$ , providing  $B$  is not negative

Proof.

Case:  $A \geq 0$ .

$$|A| = A$$

$$|A| = B$$

$$\therefore A = B$$

Case:  $A < 0$ .

$$|A| = -A$$

$$-A = B$$

$$\therefore A = -B$$

□

■  $|A| < B \implies -B < A < B$ , providing  $B$  is not negative

This says  $A$  is within  $B$  units of zero.

Proof.

Case 1:  $A \geq 0$ .

$$|A| = A$$

$$|A| < B$$

$\therefore A < B$  and  $A \geq 0$ ; that is,  $A \in [0, B)$ .

Case 2:  $A < 0$ .

$$|A| = -A$$

$$-A < B$$

$\therefore -B < A$  and  $A < 0$ ; that is,  $A \in (-B, 0)$ .

Case 1 and case 2 exhaust all cases,

$\therefore (-B < A \wedge A < 0)$  or  $(A < B \wedge A \geq 0)$ ; that is,  $a \in (-B, 0) \cup [0, B)$

Equivalently,

$\therefore -B < A < B$ ; that is,  $A \in (-B, B)$

□

■  $|A| > B \implies A < -B$  or  $A > B$

This says  $A$  is farther than  $B$  units from zero.

Proof.

Case I:  $B \geq 0$

Case 1:  $A \geq 0$ .

$$|A| = A$$

$$|A| > B$$

$$A > B$$

$\therefore 0 \leq B < A \wedge A \geq 0$ ; that is,  $A \in (B, \infty +)$ .

Case 2:  $A < 0$ .

$$|A| = -A$$

$$|A| > B$$

$$-A > B$$

$\therefore A < -B \leq 0 \wedge A < 0$ ; that is,  $A \in (-\infty, -B)$ .

$$A \in (-\infty, -B) \cup (B, \infty +)$$

Case II:  $B < 0$

Case 1:  $A \geq 0$ .

$$|A| = A$$

$$|A| > B$$

$$A > B$$

$\therefore B < 0 \wedge A > B \wedge A \geq 0 \implies A \geq 0$ ; that is,  $A \in [0, \infty +)$ .

Case 2:  $A < 0$ .

$$|A| = -A$$

$$|A| > B$$

$$-A > B$$

$\therefore A < -B \wedge 0 < -B \wedge A < 0$ ; that is,  $A \in (-\infty, 0)$ .

$$A \in (-\infty, 0) \cup [0, \infty +) \iff A \in \mathbb{R}$$

So,  $A \in (-\infty, -B) \cup (B, \infty +) \cup \mathbb{R}$

$\therefore A \in (-\infty, -B) \cup (B, \infty +)$ ; that is,  $A < -B \vee A > B$ .

□